The Decomposable Numerical Radius and Numerical Radius of a Compound Matrix

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ABSTRACT

Let A be an $n\times n$ complex matrix with singular values $\alpha_1\geqslant\alpha_2\geqslant\cdots\geqslant\alpha_n$ and eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$, where $|\lambda_1|\geqslant|\lambda_2|\geqslant\cdots\geqslant|\lambda_n|$. Denote by $C_m(A)$ $(1\leqslant m\leqslant n)$ the mth compound of A, and by $\bigwedge^m\mathbb{C}^n$ the mth Grassmann space over $\mathbb{C}^n$, in which the elements are regarded as complex row vectors with $\binom{n}{m}$ coordinates. We have the relation

$$\prod_{i=1}^{m}|\lambda_i|\leqslant r_d(C_m(A))\leqslant r(C_m(A))\leqslant\prod_{i=1}^{m}\alpha_i,$$

where

$$r_d(C_m(A))=\max\left\{|xC_m(A)x^*|: x \text{ is decomposable in } \bigwedge^m\mathbb{C}^n,\ xx^*=1\right\}$$

and

$$r(C_m(A))=\max\left\{|xC_m(A)x^*|: x\in\bigwedge^m\mathbb{C}^n,\ xx^*=1\right\}$$

are the decomposable numerical radius and numerical radius of $C_m(A)$ respectively. In this note we classify those matrices for which $\prod_{i=1}^m|\lambda_i|=r_d(C_m(A))$, $\prod_{i=1}^m|\lambda_i|=r(C_m(A))$, $r_d(C_m(A))=r(C_m(A))$, etc. Some of these results answer the questions raised by Marcus and Andresen concerning the decomposable numerical radius of a complex matrix.

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1. INTRODUCTION

Let A be an $n \times n$ complex matrix with singular values $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_n$ and eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$, where $|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n|$. Denote by $C_m(A)$ $(1 \leqslant m \leqslant n)$ the mth compound of A. Since the eigenvalues and singular values of $C_m(A)$ are the products $\prod_{i=1}^m \lambda_{\sigma(i)}$ and $\prod_{i=1}^m \alpha_{\sigma(i)}$ respectively, where $\sigma \in Q_{m,n}$ (the totality of strictly increasing sequences of m integers chosen from $1, \dots, n$), $C_m(A)$ has $\prod_{i=1}^m |\lambda_i|$ as the spectral radius and $\prod_{i=1}^m \alpha_i$ as the spectral norm. Suppose that $\overset{m}{\bigwedge} \mathbb{C}^n$ denotes the mth Grassmann space over $\mathbb{C}^n$, in which the elements are regarded as complex row vectors with $\binom{n}{m}$ coordinates, and G_m denotes the set of all unit length decomposable tensors in $\overset{m}{\bigwedge} \mathbb{C}^n$. We consider the numerical radius

$$r(C_m(A)) = \max\left\{ |xC_m(A)x^*| : x \in \overset{m}{\bigwedge} \mathbb{C}^n,\ xx^* = 1 \right\}$$

and the decomposable numerical radius

$$r_d(C_m(A)) = \max\{ |xC_m(A)x^*| : x \in G_m \}$$

of $C_m(A)$. In fact,

$$r(C_m(A)) = \max\{ |z| : z \in W(C_m(A)) \},$$

where

$$W(C_m(A)) = \left\{ xC_m(A)x^* : x \in \overset{m}{\bigwedge} \mathbb{C}^n,\ xx^* = 1 \right\}$$

is the numerical range of $C_m(A)$. Let $\mathbb{C}_{m \times n}$ and $\mathscr{U}_n$ denote the linear space of $m \times n$ complex matrices and the group of $n \times n$ unitary matrices respectively. Denote by

$$B\begin{pmatrix} \delta(1), \dots, \delta(m) \\ \sigma(1), \dots, \sigma(m) \end{pmatrix} \qquad (\delta, \sigma \in Q_{m,n})$$

the $m \times m$ submatrix of B lying in rows $\delta(1), \dots, \delta(m)$ and columns

$\sigma(1),\dots,\sigma(m)$. Marcus and Filippenko [4] have shown that

$$r_d(C_m(A)) = \max\{|z|: z \in W_m^{\wedge}(A)\},$$

where

$$\begin{aligned} W_m^{\wedge}(A) &= \{xC_m(A)x^* : x \in G_m\} \\ &= \{\det XAX^* : X \in \mathbb{C}_{m\times n}, \det XX^* = 1\} \\ &= \left\{\det UAU^*\begin{pmatrix} 1,\dots,m \\ 1,\dots,m \end{pmatrix} : U \in \mathcal{U}_n\right\} \end{aligned}$$

is the decomposable numerical range of $C_m(A)$. It is known that [3]

$$\prod_{i=1}^{m} |\lambda_i| \leqslant r_d(C_m(A)) \leqslant r(C_m(A)) \leqslant \prod_{i=1}^{m} \alpha_i.$$

Of course, if the rank of A is less than m, then all the quantities become zero and the inequalities become equalities. Aside from this trivial case, Marcus and Andresen [2, 3] have studied the matrices for which

$$\prod_{i=1}^{m} |\lambda_i| = \prod_{i=1}^{m} \alpha_i \tag{1}$$

or

$$\prod_{i=1}^{m} |\lambda_i| = r_d(C_m(A)). \tag{2}$$

Questions concerning the classification of the matrices for which (2) holds or for which

$$r_d(C_m(A)) = r(C_m(A)) \tag{3}$$

were asked in [3]. As a matter of fact, besides (1), (2), (3), we may also consider the matrices for which

$$\prod_{i=1}^{m} |\lambda_i| = r(C_m(A)), \tag{4}$$

$$r_d(C_m(A)) = \prod_{i=1}^{m} \alpha_i, \tag{5}$$

or

$$r(C_m(A)) = \prod_{i=1}^{m} \alpha_i. \tag{6}$$

In this note we classify those matrices satisfying (1), (2), (3), (4), (5), or (6) respectively.

2. RESULTS

Note that if m is equal to n, we have $W_m^{\wedge}(A) = W(C_m(A)) = \{\det A\}$ for any $n \times n$ complex matrix A. We shall always assume $1 \leqslant m \leqslant n-1$ in the following discussion unless otherwise specified.

In order to classify matrices satisfying (1), (2), (4), (5), or (6), we need the following lemmas.

LEMMA 1. *Let* $A \in \mathbf{C}_{n \times n}$. *Then* $r_d(C_m(A)) = 0$ *iff* $C_m(A) = 0$.

Proof. $\Leftarrow$: Clear

$\Rightarrow$: If $C_m(A) \neq 0$, then $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_m > 0$. Let $X, Y, U \in \mathscr{U}_n$ be such that

$$A = X \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} Y,$$

$$YX = U^* \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} U.$$

Put

$$\begin{aligned} \hat{A} &= UYAY^*U^* \\ &= UYX \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} U^* \\ &= \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} U \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} U^*. \end{aligned}$$

Then

$$\begin{aligned}
&\sum_{\sigma \in Q_{m,n}} \left| \det \hat{A} \begin{pmatrix} \sigma(1), \dots, \sigma(m) \\ \sigma(1), \dots, \sigma(m) \end{pmatrix} \right| \\
&= \sum_{\sigma \in Q_{m,n}} \det U \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} U^* \begin{pmatrix} \sigma(1), \dots, \sigma(m) \\ \sigma(1), \dots, \sigma(m) \end{pmatrix} \\
&= \sum_{\sigma \in Q_{m,n}} \alpha_{\sigma(1)} \alpha_{\sigma(2)} \cdots \alpha_{\sigma(m)} \\
&\geqslant \prod_{i=1}^{m} \alpha_i > 0.
\end{aligned}$$

Hence

$$\begin{aligned}
r_d(C_m(A)) &= r_d(C_m(\hat{A})) \\
&\geqslant \max_{\sigma \in Q_{m,n}} \left| \det \hat{A} \begin{pmatrix} \sigma(1), \dots, \sigma(m) \\ \sigma(1), \dots, \sigma(m) \end{pmatrix} \right| \\
&> 0.
\end{aligned}$$

■

LEMMA 2. *Let $A \in \mathbf{C}_{n \times n}$ with rank not less than m. If*

$$A = \left(\begin{array}{c|c} A_1 & 0 \\ \hline A_3 & A_2 \end{array} \right),$$

where $A_1 \in \mathbf{C}_{m \times m}$ and $|\det A_1| = r_d(C_m(A))$, then $A_3 = 0$. Moreover, if A is of rank m, then $A_2 = 0$ also.

Proof. Suppose that A satisfies the conditions of the lemma. By Schur's triangularization lemma (e.g. see [5, p. 158]), we can find an $m \times m$ unitary matrix U such that UA_1U^* is of the lower triangular form. Consider

$$\hat{A} = \left(\begin{array}{c|c} U & 0 \\ \hline 0 & I_{n-m} \end{array}\right)\left(\begin{array}{c|c} A_1 & 0 \\ \hline A_3 & A_2 \end{array}\right)\left(\begin{array}{c|c} U^* & 0 \\ \hline 0 & I_{n-m} \end{array}\right)$$

$$= \left(\begin{array}{c|c} UA_1U^* & 0 \\ \hline A_3U^* & A_2 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} \mu_1 & & 0 & \\ & \ddots & & 0 \\ * & & \mu_m & \\ \hline & A_3U^* & & A_2 \end{array}\right) = (a_{ij}).$$

Since the rank of A is not less than m, by Lemma 1 we have

$$\prod_{i=1}^{m} |\mu_i| = |\det A_1| = r_d(C_m(A)) > 0.$$

Consider the mth column of $\hat{A}$. If there exists an integer i such that $m < i \leqslant n$ and $a_{im} \neq 0$, then by elliptical range theorem (e.g. see [6]), we can find a 2×2 unitary matrix W such that

$$W\begin{pmatrix} \mu_m & 0 \\ a_{im} & a_{ii} \end{pmatrix}W^* = \begin{pmatrix} (1+\varepsilon)\mu_m & * \\ * & a_{ii} - \varepsilon\mu_m \end{pmatrix},$$

where $\varepsilon > 0$. Let V be the $n \times n$ unitary matrix obtained from the $n \times n$ identity matrix I by replacing $I\begin{pmatrix} m, i \\ m, i \end{pmatrix}$ with W, and let

$$V\hat{A}V^* = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right),$$

where $B_1 \in \mathbf{C}_{m \times m}$. Then

$$B_1 = \begin{pmatrix} \mu_1 & & & \\ & \ddots & & 0 \\ & & \mu_{m-1} & \\ & * & & (1+\varepsilon)\mu_m \end{pmatrix}$$

is still of the lower triangular form. As a result,

$$\begin{aligned} r_d(C_m(A)) &= r_d(C_m(\hat{A})) \\ &\geqslant |\det B_1| \\ &= |\mu_1 \cdots \mu_{m-1}(1+\varepsilon)\mu_m| \\ &> \prod_{i=1}^{m} \mu_i = r_d(C_m(A)), \end{aligned}$$

which is a contradiction. Hence $a_{im} = 0$ for $m < i \leqslant n$. Now consider the $(m-1)$th column of $\hat{A}$. If there exists an integer i such that $m < i \leqslant n$ and $a_{i,m-1} \neq 0$, then by elliptical range theorem, we can find a 2×2 unitary matrix W such that

$$W\begin{pmatrix} \mu_{m-1} & 0 \\ a_{im-1} & a_{ii} \end{pmatrix} W^* = \begin{pmatrix} (1+\varepsilon)\mu_{m-1} & * \\ * & a_{ii} - \varepsilon\mu_{m-1} \end{pmatrix},$$

where $\varepsilon > 0$. Let V be the $n \times n$ unitary matrix obtained from I by replacing $I\begin{pmatrix} m-1, i \\ m-1, i \end{pmatrix}$ with W, and let

$$V\hat{A}V^* = \left(\begin{array}{c|c} B_1 & B_2 \\ \hline B_3 & B_4 \end{array}\right),$$

where $B_1 \in \mathbf{C}_{m \times m}$. Then

$$B_1 = \begin{pmatrix} \mu_1 & & & & 0 \\ & \ddots & & & \\ & & \mu_{m-2} & & \\ * & & & (1+\varepsilon)\mu_{m-1} & \\ & & & & \mu_m \end{pmatrix}$$

is still of the lower triangular form. (Note that if we have not shown that $a_{im} = 0$ for $m < i \leqslant n$, B_1 may fail to be of the lower triangular form.) This will lead to the same contradiction as above. Therefore $a_{i,m-1} = 0$ for $m < i \leqslant n$. Under the assumption that $a_{ij} = 0$ for $m < i \leqslant n$, $j = m-1, m$, we can show that $a_{i,m-2} = 0$ for $m < i \leqslant n$ by similar arguments to the above. Inductively, we can show that $a_{ij} = 0\ (m < i \leqslant n)$ for $j = m-3,\dots,1$. As a result, $\hat{A} = UA_1U^* \oplus A_2$ and hence $A = A_1 \oplus A_2$.

Finally, if A is of rank m, then $|\det A_1| > 0$ implies that $A_2 = 0$. ∎

COROLLARY [4, Theorem 1]. *Let $1 \leqslant m < n$. A matrix $A \in \mathbf{C}_{n\times n}$ is unitary if and only if $W_m^{\wedge}(A)$ is contained in the closed unit disk and every eigenvalue of A has modulus* 1.

Proof. $\Rightarrow$: Clear.

$\Leftarrow$: If $W_m^{\wedge}(A)$ is contained in the closed unit disk, then $r_d(C_m(A)) \leqslant 1$. Suppose every eigenvalue of A has modulus 1. By Schur's triangularization lemma, A is unitarily similar to $\hat{A}$, which is of lower triangular form. For any $m \times m$ principal submatrix A_1 of $\hat{A}$, we have

$$|\det A_1| = 1 = r_d(C_m(A)) = r_d(C_m(\hat{A})).$$

By Lemma 2, $\hat{A}$ can be regarded as the direct sum of A_1 and its corresponding complement. As the $m \times m$ principal submatrix A_1 can be chosen arbitrarily, all the off-diagonal elements of $\hat{A}$ must be zero. Consequently A is normal and hence unitary. ∎

THEOREM 3. *Let $A \in \mathbf{C}_{n\times n}$ with rank not less than m. Then $\prod_{i=1}^{m}|\lambda_i| = r_d(C_m(A))$ iff A is unitarily similar to $A_1 \oplus A_2$, where $A_1 \in \mathbf{C}_{m\times m}$ and $|\det A_1| = r_d(C_m(A))$.*

Proof. $\Leftarrow$: Clear.

$\Rightarrow$: Suppose $\prod_{i=1}^{m}|\lambda_i| = r_d(C_m(A))$. By Schur's triangularization lemma, A is unitarily similar to

$$\left(\begin{array}{c|c} A_1 & 0 \\ \hline A_3 & A_2 \end{array}\right),$$

where

$$A_1 = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_m \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} \lambda_{m+1} & & 0 \\ & \ddots & \\ * & & \lambda_n \end{pmatrix}.$$

Then

$$|\det A_1| = r_d(C_m(A))$$

and by Lemma 2, the result follows. ■

THEOREM 4. *Let $A \in \mathbb{C}_{n\times n}$ with rank not less than m. Then $\prod_{i=1}^m|\lambda_i| = r(C_m(A))$ iff A is unitarily similar to $A_1 \oplus A_2$, where $A_1 \in \mathbb{C}_{m\times m}$ and $|\det A_1| = r(C_m(A))$.*

Proof. $\Leftarrow$: Clear.

$\Rightarrow$: $\prod_{i=1}^m|\lambda_i| = r(C_m(A)) \Rightarrow \prod_{i=1}^m|\lambda_i| = r_d(C_m(A)) = r(C_m(A))$. By Theorem 3, the result follows. ■

THEOREM 5. *Let $A \in \mathbb{C}_{n\times n}$ with rank not less than m. Then the following are equivalent:*

(a) *A is unitarily similar to $A_1 \oplus A_2$, where $A_1 \in \mathbb{C}_{m\times m}$ and $|\det A_1| = \prod_{i=1}^m \alpha_i$.*
(b) *$r_d(C_m(A)) = \prod_{i=1}^m \alpha_i$.*
(c) *$r(C_m(A)) = \prod_{i=1}^m \alpha_i$.*
(d) *$\prod_{i=1}^m|\lambda_i| = \prod_{i=1}^m \alpha_i$.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c): Clear.

(c) $\Rightarrow$ (d): See e.g. [1, p. 322].

(d) $\Rightarrow$ (a): $\prod_{i=1}^m|\lambda_i| = \prod_{i=1}^m \alpha_i \Rightarrow \prod_{i=1}^m|\lambda_i| = r_d(C_m(A)) = \prod_{i=1}^m \alpha_i$. By Theorem 3, the result follows. ■

THEOREM 6. *Let $A \in \mathbb{C}_{n\times n}$. Then the following are equivalent:*

(a) *A is normal.*
(b) *$r_d(C_k(A)) = \prod_{i=1}^k \alpha_i$ for $1 \leqslant k \leqslant n$.*
(c) *$r(C_k(A)) = \prod_{i=1}^k \alpha_i$ for $1 \leqslant k \leqslant n$.*
(d) *$\prod_{i=1}^k|\lambda_i| = \prod_{i=1}^k \alpha_i$ for $1 \leqslant k \leqslant n$.*
(e) *$r(C_k(A)) = \prod_{i=1}^k|\lambda_i|$ for $1 \leqslant k \leqslant n$.*
(f) *$r_d(C_k(A)) = \prod_{i=1}^k|\lambda_i|$ for $1 \leqslant k \leqslant n$.*

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d): By Theorem 5.

(d) $\Rightarrow$ (e) $\Rightarrow$ (f): Clear.

(f) $\Rightarrow$ (a): Suppose $\prod_{i=1}^k|\lambda_i| = r_d(C_k(A))$ for $1 \leqslant k \leqslant n$. By Schur's triangularization lemma, A is unitarily similar to

$$\hat{A} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ * & & \lambda_n \end{pmatrix}.$$

Then for $1 \leqslant k \leqslant n$,

$$\left|\det \hat{A}\begin{pmatrix} 1,\dots,k \\ 1,\dots,k \end{pmatrix}\right| = r_d(C_k(\hat{A})).$$

Apply Lemma 2 to $\hat{A}$ for $k = m, m-1,\dots,1$, where m is the rank of $\hat{A}$; we have

$$\hat{A} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{pmatrix} \oplus 0_{n-m},$$

and hence A is normal. ■

The following theorem concerns the matrices satisfying (3).

THEOREM 7. *Let* $A \in \mathbb{C}_{n\times n}$ *with rank not less than* m, *and* $\mathscr{C} = \{x \in \bigwedge^m \mathbb{C}^n : xx^* = 1,\ |xC_m(A)x^*| = r(C_m(A))\}$. *Then the following are equivalent*:

(a) $r_d(C_m(A)) = r(C_m(A))$.
(b) $\mathscr{C} \cap G_m \neq \varnothing$.
(c) A *is unitarily similar to*

$$\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right),$$

where $A_1 \in \mathbb{C}_{m\times m}$ *and* $|\det A_1| = r(C_m(A))$.

Moreover, if (c) *holds, then*

(i) *We have*

$$\left|\det\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)\begin{pmatrix} 1,\dots,m \\ \sigma(1),\dots,\sigma(m) \end{pmatrix}\right| = \left|\det\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)\begin{pmatrix} \sigma(1),\dots,\sigma(m) \\ 1,\dots,m \end{pmatrix}\right|$$

for any $\sigma \in Q_{m,n}$.

(ii) $A_2 = 0$ *or* $A_3 = 0$ *implies* $A_2 = A_3 = 0$ *and* $\prod_{i=1}^{m}|\lambda_i| = r(C_m(A))$.

Proof. (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c): Clear.
Suppose that (c) holds.
(i): Let $C_m(\hat{A}) = (\hat{a}_{ij}) \in \mathbf{C}_{\binom{n}{m}\times\binom{n}{m}}$, where

$$\hat{A} = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right).$$

Then $|\det A_1| = \hat{a}_{11} \in \partial W(C_m(A))$, the boundary of $W(C_m(A))$. By Lemma 2 in [8],

$$|\hat{a}_{1j}| = |\hat{a}_{j1}| \qquad \text{for} \quad 1 \leqslant j \leqslant \binom{n}{m}.$$

Thus for all $\sigma \in Q_{m,n}$,

$$\left|\det\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)\begin{pmatrix} 1,\dots,m \\ \sigma(1),\dots,\sigma(m) \end{pmatrix}\right| = \left|\det\left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right)\begin{pmatrix} \sigma(1),\dots,\sigma(m) \\ 1,\dots,m \end{pmatrix}\right|.$$

(ii): Note that $|\det A_1| = r(C_m(A))$ implies $|\det A_1| = r_d(C_m(A))$. If $A_2 = 0$, then by Lemma 2, $A_3 = 0$. By Theorem 4, $\prod_{i=1}^{m}|\lambda_i| = r(C_m(A))$. Since the quantities $r(C_m(A))$ and $\prod_{i=1}^{m}|\lambda_i|$ are invariant under transposition of A, if $A_3 = 0$, we can consider

$$\left(\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_4 \end{array}\right)^T$$

and obtain the result. ∎

3. REMARKS AND EXAMPLES

In Theorem 6, we see that if $A \in \mathbf{C}_{n\times n}$ satisfies (1), (2), (4), (5), or (6) for $m = 1,2,\dots,n$, then A is normal. The following example shows that we cannot have the same conclusion for those A satisfying (3) for $m = 1,2,\dots,n$.

EXAMPLE 1. Let

$$n \geqslant 2 \quad \text{and} \quad A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \oplus 0_{n-2}.$$

Then A is not normal, and $r_d(C_m(A)) = r(C_m(A))$ for $m = 1, \dots, n$.

In Theorem 3, Theorem 4, and Theorem 5, we see that if $A \in \mathbb{C}_{n\times n}$ satisfying (1), (2), (4), (5), or (6), then A is unitarily similar to a matrix of the form $A_1 \oplus A_2$ where $A_1 \in \mathbb{C}_{m\times m}$. The following example shows that there is no analogous result for matrices satisfying (3).

EXAMPLE 2. Suppose that $n > 2$, $1 < m < n$, and

$$A = I_{m-1} \oplus \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \oplus 0_{n-m-1}.$$

Then $\prod_{i=1}^{m} |\lambda_i| = 0$. Moreover,

$$C_m(A) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \oplus 0_{\binom{n}{m}-2}$$

and hence $r(C_m(A)) = 1$. As A is unitarily similar to

$$\hat{A} = I_{m-1} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \oplus 0_{n-m-1},$$

we have

$$\det \hat{A}\begin{pmatrix} 1, \dots, m \\ 1, \dots, m \end{pmatrix} = 1.$$

Consequently,

$$\begin{aligned} r(C_m(A)) &= r_d(C_m(A)) \\ &= 1 \\ &> 0 \\ &= \prod_{i=1}^{m} |\lambda_i|. \end{aligned}$$

By Theorem 7(ii), A must not be unitarily similar to $A_1 \oplus A_2$ with $A_1 \in \mathbf{C}_{m\times m}$ and $|\det A_1| = r(C_m(A))$.

Since $r(C_m(A)) = \max\{|z|: z \in W(C_m(A))\}$, $r_d(C_m(A)) = \max\{|z|: z \in W_m^{\wedge}(A)\}$, and

$$\prod_{i=1}^{m} a_i = \max\{|z|: z \in D(C_m(A))\},$$

where

$$D(C_m(A)) = \left\{xC_m(A)y^* : x, y \in \bigwedge^m \mathbf{C}^n,\ xx^* = yy^* = 1\right\},$$

one might expect to be able to define a new quantity for A by the expression

$$\max\{|z|: z \in D_m^{\wedge}(A)\},$$

where

$$\begin{aligned} D_m^{\wedge}(A) &= \{xC_m(A)y^* : x, y \in G_m\} \\ &= \{\det XAY^* : X, Y \in \mathbf{C}_{m\times n},\ \det XX^* = \det YY^* = 1\} \\ &= \left\{\det UAV\begin{pmatrix} 1,\dots,m \\ 1,\dots,m \end{pmatrix} : U, V \in \mathscr{U}_n\right\}. \end{aligned}$$

However, we have the following.

THEOREM 8. *Let $A \in \mathbf{C}_{n\times n}$ and $1 \leqslant m \leqslant n$. Then $D(C_m(A)) = D_m^{\wedge}(A)$.*

Proof. If $n = m \geqslant 1$, then $D(C_m(A)) = \{e^{i\theta}\det A : \theta \in \mathbb{R}\} = D_m^{\wedge}(A)$. If $n > m \geqslant 1$, then by a result of von Neumann [7],

$$D(C_m(A)) = \left\{z : |z| \leqslant \prod_{i=1}^{m} \alpha_i\right\}.$$

As $D_m^{\wedge}(A) \subset D(C_m(A))$, we have only to prove

$$\left\{z : |z| \leqslant \prod_{i=1}^{m} \alpha_i\right\} \subset \left\{\det UAV\begin{pmatrix} 1,\dots,m \\ 1,\dots,m \end{pmatrix} : U, V \in \mathscr{U}_n\right\}.$$

Since the set $D_m^{\wedge}(A)$ has circular symmetry about the origin, the function $f\colon \mathscr{U}_n \times \mathscr{U}_n \to \mathbb{C}$ defined by

$$f(U,V) = \det UAV\begin{pmatrix} 1,\ldots,m \\ 1,\ldots,m \end{pmatrix}$$

is continuous, and $\mathscr{U}_n \times \mathscr{U}_n$ is connected and compact, we can conclude that $D_m^{\wedge}(A)$ is an annulus on $\mathbb{C}$ centered at the origin. Let $U_0, V_0 \in \mathscr{U}_n$ be such that

$$A = U_0^* \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} V_0^*.$$

Then $f(U_0, V_0) = \prod_{i=1}^m \alpha_i$ and $f(PU_0, V_0) = 0$, where P is the permutation matrix obtained from I by interchanging the first and the last row. Hence $\{ z : |z| \leqslant \prod_{i=1}^m \alpha_i \} \subset D_m^{\wedge}(A)$. As a result, $D(C_m(A)) = D_m^{\wedge}(A)$. ■

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